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Original Research Paper

Solution of Linear Systems with MAXIMA

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The methods of calculus lie at the heart of the physical sciences and engineering. Maxima can help you make faster progress, if you are just learning calculus. The examples in this research paper will offer an opportunity to see some Maxima tools in the context of simple examples, but you will likely be thinking about much harder problems you want to solve as you see these tools used here. This research paper includes Solution of Linear Systems with MAXIMA.

Keywords: Linear system, Gauss elimination, LU Decomposition, Gauss-Jordan elimination, Jacobi iteration method, Gauss-Seidel method, Conjugate Gradient method, Maxima.

1. INTRODUCTION

Maxima is a system for the manipulation of symbolic and numerical expressions, including differentiation, integration, Taylor series, Laplace transforms, ordinary differential equations, systems of linear equations, polynomials, sets, lists, vectors, matrices, tensors, and more. Maxima yields high precision numeric results by using exact fractions, arbitrary precision integers, and variable precision floating point numbers. Maxima can plot functions and data in two and three dimensions.

Maxima source code can be compiled on many computer operating systems, including Windows, Linux, and MacOS X. The source code for all systems and precompiled binaries for Windows and Linux are available at the SourceForge file manager.

Maxima is a descendant of Macsyma, the legendary computer algebra system developed in the late 1960s at the Massachusetts Institute of Technology. It is the only system based on that effort still publicly available and with an active user community, thanks to its open source nature. Macsyma was revolutionary in its day, and many later systems, such as Maple and Mathematica, were inspired by it.

The Maxima branch of Macsyma was maintained by William Schelter from 1982 until he passed away in 2001. In 1998 he obtained permission to release the source code under the GNU General Public License (GPL). It was his efforts and skills that made the survival of Maxima possible.
MAXIMA has been constantly updated and used by researcher and engineers as well as by students.

2. SOLUTION OF LINEAR SYSTEMS

$$\mathbf{Ax} = \mathbf{b}$$

2.1. Introduction

The linear system has the form

$$\mathbf{Ax} = \mathbf{b}$$

where **A** is the coefficient matrix, **x** is unknown vector, and **b** is known vector.

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There are two methods used to solve systems of linear and algebraic equations. They are;

2.1.1. Direct methods

Gauss elimination, LU Decomposition, Gauss-Jordan elimination.

2.1.2. Iterative methods: (indirect methods)

Jacobi iteration method, Gauss-Seidel method, Conjugate Gradient method.

2.2. Gauss Elimination Method

Gauss elimination is the most familiar method based on a row by row elimination. The function of the elimination is to transform the equations into the form **Ux = c**, where **U** is an upper triangular matrix. **The rule of elimination:** Multiply one equation (pivot equation) by a constant and subtract it from another equation. Then, the equations are solved by using the back substitution.

Example 2.1.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.1)$$
$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.2)$$
$$4x_1 - x_2 + x_3 = 5 \quad (2.3)$$

Find the x_1, x_2 *and* x_3 by using Gauss Elimination Method.

Solution:

Take Eq.(2.1) as pivot equation, x_1 is eliminated by subtracting -2 x Eq.(2.1) from Eq.(2.2) as

$$\left.\begin{array}{l} Eq.(2.2): \quad \to -2x_1 + 5x_2 + x_3 = 11 \\ -2xEq.(2.1) \to -2x_1 + 4x_2 - 10x_3 = -24 \end{array}\right\} x_2 + 11x_3 = 35$$

Subtract 4 x Eq.(2.1) from Eq.(2.3) as

$$\left.\begin{array}{l} \text{Eq.(2.3):} \quad \to 4x_1 - x_2 + x_3 = 5 \\ 4xEq.(2.1) :\to 4x_1 - 8x_2 + 20x_3 = 48 \end{array}\right\} 7x_2 - 19x_3 = -43$$

Therefore;

$$x_2 + 11x_3 = 35 \quad (2.4)$$
$$7x_2 - 19x_3 = -43 \quad (2.5)$$

The same procedure is applied for Eqs.(2.4) and (2.5) by taking Eq.(2.4) as pivot equation, x_2 is eliminated by subtracting
7 x Eq.(2.4) from Eq.(2.5) as

$$\text{Eq.(2.5):} \quad \to 7x_2 - 19x_3 = -43$$
$$7xEq.(2.4) :\to 7x_2 + 77x_3 = 245$$

Therefore

$$-96x_3 = -288 \quad (2.6)$$
$$x_3 = \frac{-288}{-96} = 3.$$

Putting x_3 in Eq. (2.4) gives

$$x_2 + 11x_3 = 35 \to x_2 = 35 - 33 = 2.$$

Then, by using x_2 and x_3 in Eq.(2.1) gives

$$x_1 - 2x_2 + 5x_3 = 12 \to x_1 = 12 + 4 - 15 = 1.$$

Writing back substitution in terms of **Ux = c** by using Eqs.(2.1), (2.4), (2.6)

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.1)$$
$$x_2 + 11x_3 = 35 \quad (2.4)$$
$$-96x_3 = -288 \quad (2.6)$$
$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 11 \\ 0 & 0 & -96 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 35 \\ -288 \end{bmatrix} \quad (2.7)$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The determinant of the coefficient matrix

$$|\mathbf{A}| = \begin{vmatrix} 1 & -2 & 5 \\ -2 & 5 & 1 \\ 4 & -1 & 1 \end{vmatrix} = -96$$

is same as the determinant of the upper triangular matrix $|\mathbf{U}|$.

2.2.1. Pivoting

If the diagonal component of the coefficient matrix is zero or become zero during the elimination, there will a problem. We can solve this problem by interchanging row. The algorithm of pivoting provides the following:

- Leading diagonal element
- Largest coefficient in the rows
- Moving it into the leading diagonal position
- Factorization (elimination)

Example 2.2.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.8)$$
$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.9)$$
$$4x_1 - x_2 + x_3 = 5 \quad (2.10)$$

Find x_1, x_2 *and* x_3 by using pivoting.

Solution:
Examine the values of the first column $|1|, |-2|$ *and* $|4|$. The largest absolute value is 4. So

$$4x_1 - x_2 + x_3 = 5 \quad (2.11)$$
$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.12)$$
$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.13)$$

$$\left.\begin{array}{l} \text{Eq.}(2.12) \rightarrow -2x_1 + 5x_2 + x_3 = 11 \\ -\frac{1}{2}\text{Eq.}(2.11) \rightarrow -2x_1 + 0.5x_2 - 0.5x_3 = -2.5 \end{array}\right\} 4.5x_2 + 1.5x_3 = 13.5$$

$$\left.\begin{array}{l} \text{Eq.}(2.13) \rightarrow x_1 - 2x_2 + 5x_3 = 12 \\ \frac{1}{4}\text{Eq.}(2.11) \rightarrow x_1 - 0.25x_2 + 0.25x_3 = 1.25 \end{array}\right\} -1.75x_2 + 4.75x_3 = 10.75$$

Therefore;

$$4.5x_2 + 1.5x_3 = 13.5 \quad (2.14)$$
$$-1.75x_2 + 4.75x_3 = 10.75 \quad (2.15)$$

$$\left.\begin{array}{l} \text{Eq.}(2.15) \rightarrow -1.75x_2 + 4.75x_3 = 10.75 \\ -\frac{7}{18}\text{Eq.}(2.14) \rightarrow -1.75x_2 - \frac{7}{12}x_3 = -5.25 \end{array}\right\} \frac{16}{3}x_3 = 16$$

Therefore;

$$\frac{16}{3}x_3 = 16 \quad (2.16)$$
$$x_3 = 3.$$

Putting x_3 in Eq.(2.15) gives

$$-1.75x_2 + 4.75x_3 = 10.75 \rightarrow x_2 = \frac{10.75 - 14.25}{-1.75} = 2.$$

By using x_2 and x_3 in Eq.(2.12) gives

$$-2x_1 + 5x_2 + x_3 = 11 \rightarrow x_1 = \frac{11 - 3 - 10}{-2} = 1.$$

Find the result:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.3. LU Decomposition

LU decomposition, modified form of Gaussian elimination, is a matrix decomposition with the product of lower triangular matrix (**L**) and upper triangular matrix (**U**) of the form

$$\mathbf{A} = \mathbf{LU} \quad (2.17)$$

where **L** has zeros above the diagonal and **U** has zeros below the diagonal of a square matrix **A**. For 3 x 3 **A** matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (2.18)$$

either the diagonal of **L** or the diagonal of **U** can be taken as unity for Doolittle's decomposition and Crout's decomposition respectively.

For Doolittle's decomposition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \quad (2.19)$$

For the linear system $\mathbf{Ax} = \mathbf{b} \rightarrow \mathbf{L(Ux)} = \mathbf{b}$ with forward substitution in terms of **Lc = b** for **c**, and back substitution in terms of **Ux = c** for **x**, the linear system is solved.

Example 2.3.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.20)$$
$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.21)$$
$$4x_1 - x_2 + x_3 = 5 \quad (2.22)$$

Find x_1, x_2 *and* x_3 by using the LU Doolittle's decomposition.

Solution:

The coefficient matrix is factorized with lower and upper triangular matrices

$$\begin{bmatrix} 1 & -2 & 5 \\ -2 & 5 & 1 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (2.23)$$

The upper and lower triangular matrices are solved.

$$\left.\begin{array}{l} 1.u_{11} = 1 \\ 1.u_{12} = -2 \\ 1.u_{13} = 5 \end{array}\right\} \Rightarrow \begin{array}{l} u_{11} = 1 \\ u_{12} = -2 \\ u_{13} = 5 \end{array} \quad (2.24)$$

$$\left.\begin{array}{l} l_{21}.u_{11} = -2 \\ l_{21}.u_{12} + u_{22} = 5 \\ l_{21}.u_{13} + u_{23} = 1 \end{array}\right\} \Rightarrow \begin{array}{l} l_{21} = -2 \\ u_{22} = 5 - 4 = 1 \\ u_{23} = 1 + 10 = 11 \end{array} \quad (2.25)$$

$$\left.\begin{array}{l} l_{31}.u_{11} = 4 \\ l_{31}.u_{12} + l_{32}.u_{22} = -1 \\ l_{31}.u_{13} + l_{32}.u_{23} + u_{33} = 1 \end{array}\right\} \Rightarrow \begin{array}{l} l_{31} = 4 \\ l_{32} = -1 + 8 = 7 \\ u_{33} = 1 - 20 - 77 = -96 \end{array} \quad (2.26)$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 7 & 1 \end{bmatrix} \quad (2.27)$$

$$\mathbf{U} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 11 \\ 0 & 0 & -96 \end{bmatrix} \quad (2.28)$$

Forward substitution:

$$\mathbf{Lc = b} \quad (2.29)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 7 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \\ 5 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 12 \\ 35 \\ -288 \end{bmatrix} \quad (2.30)$$

Back substitution:

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 11 \\ 0 & 0 & -96 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 35 \\ -288 \end{bmatrix} \quad (2.32)$$

Find solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.4. Gauss-Jordan Elimination Method

The Gauss-Jordan method is an elimination method by putting zeros above and below each pivot element. It is generally used for matrix inverse calculation. The steps of the Gauss-Jordan elimination are

The augmented matrix **[A b]** is used.

With the help of row operations (interchanging, scaling, replacement), the identity matrix is tried to be obtained.

Example 2.4.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.33)$$

$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.34)$$

$$4x_1 - x_2 + x_3 = 5 \quad (2.35)$$

Find x_1, x_2 *and* x_3 by using the Gauss-Jordan Elimination Method.

Solution:

The augmented matrix for the above example is

$$\begin{bmatrix} 1 & -2 & 5 & 12 \\ -2 & 5 & 1 & 11 \\ 4 & -1 & 1 & 5 \end{bmatrix} \quad (2.36)$$

The third row is moved to the first row considering pivoting

$$\begin{bmatrix} 4 & -1 & 1 & 5 \\ 1 & -2 & 5 & 12 \\ -2 & 5 & 1 & 11 \end{bmatrix} \quad (2.37)$$

Multiply first row by -1/4 and add to the second row

$$\begin{bmatrix} 4 & -1 & 1 & 5 \\ 0 & -1.75 & 4.75 & 10.75 \\ -2 & 5 & 1 & 11 \end{bmatrix} \quad (2.38)$$

Multiply first row by 1/2 and add to the third row

$$\begin{bmatrix} 4 & -1 & 1 & 5 \\ 0 & -1.75 & 4.75 & 10.75 \\ 0 & 4.5 & 1.5 & 13.5 \end{bmatrix} \quad (2.39)$$

The third row is moved to the second row considering pivoting

$$\begin{bmatrix} 4 & -1 & 1 & 5 \\ 0 & 4.5 & 1.5 & 13.5 \\ 0 & -1.75 & 4.75 & 10.75 \end{bmatrix} \quad (2.40)$$

Multiply second row by 2/9 and add to the first row

$$\begin{bmatrix} 4 & 0 & 1.33 & 8 \\ 0 & 4.5 & 1.5 & 13.5 \\ 0 & -1.75 & 4.75 & 10.75 \end{bmatrix} \quad (2.41)$$

Multiply second row by 7/18 and add to the third row

$$\begin{bmatrix} 4 & 0 & 1.33 & 8 \\ 0 & 4.5 & 1.5 & 13.5 \\ 0 & 0 & 5.33 & 16 \end{bmatrix} \quad (2.42)$$

Multiply third row by -1/4 and add to the first row

$$\begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 4.5 & 1.5 & 13.5 \\ 0 & 0 & 5.33 & 16 \end{bmatrix} \quad (2.43)$$

Multiply third row by -9/32 and add to the second row

$$\begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5.33 & 16 \end{bmatrix} \quad (2.44)$$

Multiply first, second and third row by 1/4, 2/9, and 3/16 respectively

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad (2.45)$$

The last column gives the solution vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.5. Jacobi Iteration Method

The Jacobi method is an iterative method, which means guessing the solution and then improving the solution iteratively.

The necessary condition to use this method is that the diagonal has no zeros. The coefficient matrix (**A**) can be written as a summation of diagonal, lower and upper triangular parts of it. It is important that the triangular matrices have no relation with the matrices **L** and **U** used in the previous section (direct methods). They are only additive components of **A**. So, the solution of linear equations can be shown

$$\mathbf{Ax = b} \quad (2.46)$$

$$\mathbf{(I + L + U)x = b} \quad (2.47)$$

$$\mathbf{x = b - (L + U)x} \quad (2.48)$$

As a result, for the ith equation of system,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (2.49)$$

while considering $a_{ii} \neq 0, \quad the \quad (k+1)^{th}$ iteration solution can be written

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right) \quad (2.50)$$

The steps of the Jacobi iteration method are Guess initial solution $x^{(0)}$.

Divide each equation by the diagonal term.

Find $S_i = \sum a_{ij} x_j^{(k)} \quad for \quad j \neq i.$

Find the iterated solution $x_i^{(k+1)} = \frac{b_i - S_i}{a_{ii}}.$

Repeat for n equation and find x_i for i = 1,2,…:n. Go to the third step for the next iteration using the new iterated solution.

Example 2.5.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.51)$$

$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.52)$$

$$4x_1 - x_2 + x_3 = 5 \quad (2.53)$$

Find $x_1, x_2 \quad and \quad x_3$ by using the Jacobi Iteration Method.

Solution:

Divide all coefficients by the diagonal term, then find **L** and **U**

$$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = \frac{5}{4} \quad (2.54)$$

$$-\frac{2}{5}x_1 + x_2 + \frac{2}{5}x_3 = \frac{11}{5} \quad (2.55)$$

$$\frac{1}{5}x_1 - \frac{2}{5}x_2 + x_3 = \frac{12}{5} \quad (2.56)$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{2}{5} & 0 & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix} \quad (2.57)$$

and

$$\mathbf{U} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \quad (2.58)$$

The iteration process equation for k-step to the (k+1) step

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} \frac{5}{4} \\ \frac{11}{5} \\ \frac{12}{5} \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ -\frac{2}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k} \quad (2.59)$$

Calculation:

Guess $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ to start.

Results of the iteration are shown in Table 2.1

Final result:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.6. Gauss-Seidel Method

For Jacobi Iteration Method, all **x** are modified for the next iteration. For Gauss-Seidel Method, the new component of **x** is immediately evaluated before all of linear equations are completed. Same as before, the condition that the diagonal has no zeros has been considered. The coefficient matrix (**A**) can be written as **A = I + L + U** in terms of diagonal, lower and upper triangular parts of itself. So, the solution of linear equations can be shown

$$\mathbf{Ax = b} \quad (2.60)$$

$$\mathbf{(I + L + U)x = b} \quad (2.61)$$

$$\mathbf{x = (I + L)^{-1}\left[b - Ux\right]} \quad (2.62)$$

As a result, while considering $a_{ii} \neq 0,$ *the* $(k+1)^{th}$ iteration solution can be written

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right) \quad (2.63)$$

The steps of the Gauss-Seidel Method are Guess initial solution $x^{(0)}$.

Divide each equation by the diagonal term.

Find $S_{i1} = \sum_{j<i} a_{ij}x_j^{(k+1)} \quad for \quad j < i.$

Find $S_{i2} = \sum_{j<i} a_{ij}x_j^{(k)} \quad for \quad j > i.$

Find $S_i = S_{i1} + S_{i2}$.

Find the iterated solution $x_i^{(k+1)} = \dfrac{b_i - S_i}{a_{ii}}.$

Repeat for n equation and find xi for i = 1, 2,…n.

Go to the third step for the next iteration using the new iterated solution.

Example 2.6.

Consider a set of equations:

$$x_1 - 2x_2 + 5x_3 = 12 \quad (2.64)$$

$$-2x_1 + 5x_2 + x_3 = 11 \quad (2.65)$$

$$4x_1 - x_2 + x_3 = 5 \quad (2.66)$$

Find x_1, x_2 *and* x_3 by using the Gauss-Seidel Method.

Solution:

Divide all coefficients by the diagonal term, then

$$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = \frac{5}{4} \quad (2.67)$$

$$-\frac{2}{5}x_1 + 2x_2 + \frac{1}{5}x_3 = \frac{11}{5} \quad (2.68)$$

$$\frac{1}{5}x_1 - \frac{2}{5}x_2 + x_3 = \frac{12}{5} \quad (2.69)$$

Table 2.1: Results of iteration, obtained after each step

k	x_1	x_2	x_3
1	0.0000	0.0000	0.0000
2	1.2500	2.2000	2.4000
3	1.2000	2.2200	3.0300
4	1.0475	2.0740	3.0480
5	1.0065	2.0094	3.0201
6	0.9973	1.9986	3.0025
7	0.9990	1.9984	3.0000
8	0.9996	1.9996	2.9996
9	1.0000	1.9999	2.9999
10	1.0000	2.0000	3.0000

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{2}{5} & 0 & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix} \quad and \quad \mathbf{U} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} \frac{5}{4} \\ \frac{11}{5} \\ \frac{12}{5} \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k} \quad (2.70)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ \frac{1}{5} & -\frac{2}{5} & 0 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \frac{5}{4} \\ \frac{11}{5} \\ \frac{12}{5} \end{bmatrix} - \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k} \right\}_k \quad (2.71)$$

Iteration process:

Guess $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ to start. Use forward substitution

Results of iteration, obtained after each step are shown in Table 2.2.

Find result:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.7. Conjugate Gradient Method

If the coefficients matrix is symmetric

$$\left(\mathbf{A} = \mathbf{A}^{\mathbf{T}}\right)$$

and positive definite

$$\left(\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} > 0\right),$$

the minimization of quadratic function with the form

$$\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} - \mathbf{b}^{\mathbf{T}}\mathbf{x}$$

can be considered for the solution.

The steps of the Conjugate gradient Method are as follows:

Guess initial solution

$$\mathbf{x}^{(0)}$$

.
Find the initial residual

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)}$$

.
The initial search direction (descent vector) is taken same as the initial residual

$$\mathbf{s}^{(0)} = \mathbf{r}^{(0)}$$

Find the iterated solution

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{\left(\mathbf{r}^{(k)}\right)^T \mathbf{r}^{(k)}}{\left(\mathbf{s}^{(k)}\right)^T \mathbf{A}\mathbf{s}^{(k)}} \mathbf{s}^{(k)}$$

Find the iterated residual

$$\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} + \frac{\left(\mathbf{r}^{(k)}\right)^T \mathbf{r}^{(k)}}{\left(\mathbf{s}^{(k)}\right)^T \mathbf{A}\mathbf{s}^{(k)}} \mathbf{A}\mathbf{s}^{(k)}$$

Table 2.2: Results of iteration, obtained after each step

k	x_1	x_2	x_3
1	0.0000	0.0000	0.0000
2	1.2500	2.7000	3.2300
3	1.1175	2.0010	2.9769
4	1.0060	2.0070	3.0016
5	1.0014	2.0002	2.9998
6	1.0001	2.0001	3.0000
7	1.0000	2.0000	3.0000

Find the iterated search direction

$$\mathbf{s}^{(k+1)} = \mathbf{r}^{(k)} + \frac{\left(\mathbf{r}^{(k)}\right)^T \mathbf{r}^{(k)}}{\left(\mathbf{r}^{(k)}\right)^T \mathbf{r}^{(k)}} \mathbf{s}^{(k)}$$

Go to the fourth step for the next iteration using the new iterated solution.

Example 2.7.

Consider a set of equations:

$$5x_1 - x_2 + 2x_3 = 9 \quad (2.72)$$

$$-x_1 + 4x_2 - 3x_3 = -2 \quad (2.73)$$

$$2x_1 - 3x_2 + 5x_3 = 11 \quad (2.74)$$

Find x_1, x_2 *and* x_3 by using the Conjugate Gradient Method.

Solution:

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 4 & -3 \\ 2 & -3 & 5 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 9 \\ -2 \\ 11 \end{bmatrix}$$

Guess $\mathbf{x}^{(0)} = [000]^T$ to start.

The initial residual $\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} 9 \\ -2 \\ 11 \end{bmatrix}.$

The initial descent vector $\mathbf{s}^{(0)} = \begin{bmatrix} 9 \\ -2 \\ 11 \end{bmatrix}.$

The iteration results are shown in Table 2.3 below.

Find result:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.7. Stabilized Bi-conjugate Gradient Method

If the coefficient matrix is not symmetric, the Stabilized Bi-conjugate Gradient Method can be considered. The steps of this method can be summarized as follows

Guess initial solution $x^{(0)}$.

Find the initial residual $\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)}$.

The initial values for

$$\mathbf{v}^{(0)} = \mathbf{p}^{(0)} \quad and \quad c_1^{(0)} = \left(\mathbf{r}^{(0)}\right)^T \mathbf{r}^{(0)}, \quad c_0^{(0)} = \omega^{(0)} = \alpha^{(0)} = 1.$$

Find

$$\mathbf{p}^{(k+1)} = \mathbf{r}^{(k)} + \frac{c_1^{(k)}\alpha^{(k)}}{c_0^{(k)}\omega^{(k)}}(\mathbf{p}^{(k)} - \omega^{(k)}\mathbf{v}^{(k)}) \quad and \quad \mathbf{v}^{(k+1)} = \mathbf{A}\mathbf{p}^{(k+1)}$$

Find $\alpha^{(k+1)} = \dfrac{c_1^{(k)}}{(\mathrm{r}^{(0)})^T \mathbf{v}^{(k+1)}}.$

Find $\mathbf{s}^{(k+1)} = \mathbf{r}^{(k)} - \alpha^{(k+1)}\mathbf{v}^{(k+1)}.$

Find $\mathbf{t}^{(k+1)} = \mathbf{A}\mathbf{s}^{(k+1)}.$

Find $\omega^{(k+1)} = \dfrac{(\mathrm{t}^{(k+1)})^T \mathbf{s}^{(k+1)}}{(\mathrm{t}^{(k+1)})^T t^{(k+1)}}.$

Find $c_0^{(k+1)} = c_1^{(k)}$ *and* $c_1^{(k+1)} = -\omega^{(k+1)}(\mathbf{r}^{(0)})^T t^{(k+1)}.$

Find the iterated solution

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k+1)}\mathbf{p}^{(k+1)} + \omega^{(k+1)}\mathbf{s}^{(k+1)}.$$

Table 2.3: Results of iteration, obtained after each step

k	x_1	x_2	x_3	r	S
1	0.0000	0.000	0.0000	$[9,-2,11]^T$	$[9,-2,11]^T$
2	1.1660	-0.251	21.42	$[0.06,4.48,0.77]^T$	$[0.96,4.28,1^T$
3	1.5951	1.645	2.257	$[1.84,0.23,1.47]^T$	$[-1.58,0.93,1.97]^T$
4	1.0000	2.000	3.000	$[0.00,0.00,0.00]^T$	$[0.00,0.00,0^T$

Find the iterated residual $\mathbf{r}^{(k+1)} = \mathbf{s}^{(k+1)} - \omega^{(k+1)}\mathbf{t}^{(k+1)}$.

Go to the fourth step for the next iteration using the new iterated solution.

Example 2.8.

Consider a set of equations:

$$5x_1 - x_2 + 2x_3 = 9 \quad (2.72)$$

$$-x_1 + 4x_2 - 3x_3 = -2 \quad (2.73)$$

$$2x_1 - 3x_2 + 5x_3 = 11 \quad (2.74)$$

Find x_1, x_2 *and* x_3 by using the Conjugate Gradient Method.

Solution:

Guess $x^{(0)} = [0\ 0\ 0]^T$ to start

The iteration results are shown in Table 2.4 below.

Find result:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.8. MAXIMA Application

2.8.1. Introduction

The solution of simultaneous equations occurs frequently in applications of engineering mathematics. Nonlinear equations arise in a number of instances, such as optimization, eigenvalue calculations, simulations, etc.

2.8.2. Examples

1.

```
(%i1) eq:x^2-1;
      solve(eq,x);
(%o1) x^2-1
(%o2) [x=-1,x=1
```

2.

```
(%i3) eq:tan(x)-a*sin(x)=a$
      solve(eq,x);
```

$$\text{(\%o4)}\quad [\sin(x) = -\frac{a\cos(x)}{a\cos(x)-1}]$$

3.

```
(%i5) eq:a*sin(cos(x)*(f(x)-1));
      solve(eq,x);
(%o5) a sin((f(x)-1) cos(x))
solve: using arc-trig functions to get a solution
Some solutions will be lost.
```

$$\text{(\%o6)}\quad [x=\frac{\pi}{2}, f(x)=1]$$

4.

```
(%i7) eq:a*sin(x)-x/b;
      solve(eq,x);
```

$$\text{(\%o7)}\quad a\sin(x) - \frac{x}{b}$$

$$\text{(\%o8)}\quad [x=a\,b\sin(x)]$$

5.

```
(%i9) eq1:x+y+z=1$
      eq2:x-y+z=2$
      eq3:x-y-z=3$
      solve([eq1,eq2,eq3],[x,y,z]);
```

$$\text{(\%o12)}\quad [[x=2, y=-\frac{1}{2}, z=-\frac{1}{2}]]$$

6.

```
(%i13) solve([x+y=1,2*x+2*y=2],[x,y]);
solve: dependent equations eliminated: (2)
(%o13) [[x=1-%r1,y=%r1]]
```

7.

```
(%i14) eq:(n+1)*f(n)-(n-1)*f(n-1)=(n+1)/(n-1);
       funcsolve(eq,f(n));
```

$$\text{(\%o14)}\quad (n+1)\,f(n) - (n-1)\,f(n-1) = \frac{n+1}{n-1}$$

$$\text{(\%o15)}\quad f(n) = \frac{\%0\,n^2 + \%1\,n + \%2}{(n-1)\,(n+1)}$$

8.

```
(%i16) eq1:x+y+z=a$
       eq2:x-y+z=b$
       eq3:x+y-z=c$
       solve([eq1,eq2,eq3],[x,y,z]);
       linsolve([eq1,eq2,eq3],[x,y,z]);
```

$$\text{(\%o19)}\quad [[x=\frac{c+b}{2}, y=\frac{a-b}{2}, z=\frac{a-c}{2}]]$$

$$\text{(\%o20)}\quad [x=\frac{c+b}{2}, y=\frac{a-b}{2}, z=\frac{a-c}{2}]$$

Table 2.4: Results of iteration, obtained after each step

k	x_1	x_2	x_3
1	0.0000	0.0000	0.0000
2	1.1807	2.2039	3.1035
3	1.0178	2.0201	3.0105
4	1.0000	2.0000	3.0000

9.

```
(%i17) eq1:4*x-y+z=5$
       eq2:-2*x+5*y+z=11$
       eq3:x-2*y+5*z=12$
       solve([eq1,eq2,eq3],[x,y,z]);
(%o20) [[x=1,y=2,z=3]]
```

3. CONCLUSION

This research paper can apply each and every part of Solution of Linear Systems, help applications of both the physical sciences and engineering, make faster progress, and help to understand Solution of Linear Systems faster. The paper particularity helps to understand parts of Linear Algebra and is going to extend to other parts of the Linear Algebra.

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